PUBLISHED BY INSTITUTE OF PHYSICS PUBLISHING FOR SISSA

RECEIVED: *March 7, 2008*
ACCEPTED: *April 25, 2008*
PUBLISHED: *May 14, 2008*

Quantum giant magnons

K. Zarembo*

Department of Physics and Astronomy, Uppsala University
SE-751 08 Uppsala, Sweden, and
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure
24 rue Lhomond, Paris CEDEX 75231, France
E-mail: Konstantin.Zarembo@teorfys.uu.se

ABSTRACT: The giant magnons are classical solitons of the $O(N)$ sigma-model, which play an important role in the AdS/CFT correspondence. We study quantum giant magnons first at large N and then exactly using Bethe Ansatz, where giant magnons can be interpreted as holes in the Fermi sea. We also identify a solvable limit of Bethe Ansatz in which it describes a weakly-interacting Bose gas at zero temperature. The examples include the $O(N)$ model at large-N, weakly interacting non-linear Schrödinger model, and nearly isotropic XXZ spin chain in the magnetic field.

KEYWORDS: Sigma Models, 1/N Expansion, Integrable Field Theories, Bethe Ansatz.

*Also at ITEP, Moscow, Russia

JHEP05(2008)047

Contents

1. Introduction

The giant magnons [1] are solitons on the string world-sheet in $AdS_5 \times S^5$ and are argued to be the fundamental building blocks of the spectrum in the AdS/CFT correspondence. One of the remarkable features of giant magnons is periodicity of their momentum, which has a geometric origin [1]. This periodicity is quite puzzling since the centre of mass of a giant magnon, the collective coordinate canonically conjugate to the momentum, should then be quantized, pointing perhaps to some underlying lattice structure in the sigma model on $AdS_5 \times S^5$. The semiclassical quantization of the giant magnon was carried out in [2–6]. The purpose of this paper is to go beyond the semiclassical approximation, albeit not in string theory in $AdS_5 \times S^5$. The prime example will be the $O(N)$ sigma-model, which also admits giant magnons as classical solutions. Following [7], we will identify quantum giant magnons in the $O(N)$ model with the holes in the Fermi sea of the fundamental vector particles. The Fermi sea arises in the exact Bethe-Ansatz solution of the model [8–11].

The analogy with [7] is possible because the solitons of the non-linear Schrödinger (NLS) equation considered there have much in common with giant magnons. Both are particular examples of *dark solitons* [12]. A dark soliton can be pictured as a dark spot moving through a bright medium (hence the name) or, more appropriately in the present context, as a localized dilution of the Bose-Einstein condensate. It is characterized by two conditions: (i) finite background density: $\phi \to \phi\, \mathrm{e}^{-i\mu t}$, $\langle \phi \rangle \neq 0$ (ϕ is the field that carries the condensed charge, μ is the chemical potential);[1] and (ii) twisted boundary conditions: $\phi(+\infty,t) = \mathrm{e}^{i\Delta\varphi}\phi(-\infty,t)$. In other words, the phase of ϕ experiences a finite increment as one crosses the soliton, and the modulus of ϕ has a dip in the soliton's core.

We will consider giant magnon solutions (which belong to the class of dark solitons described above) in the $O(N)$ sigma model:

$$S = \frac{N}{2\lambda}\int d^2x\,\partial^\nu \mathbf{n}\cdot\partial_\nu \mathbf{n}, \tag{1.1}$$

where $\mathbf{n}$ is an N-dimensional unit vector. The giant magnon is a soliton on S^2, which in terms of the charged field $\phi = \sin\vartheta\,\mathrm{e}^{i\varphi}$, where $\mathbf{n} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta, \mathbf{0})$, has the form [1]

$$\phi = \mathrm{e}^{-i\mu t}\left(v - i\sqrt{1-v^2}\tanh\frac{\mu(x-vt)}{\sqrt{1-v^2}}\right). \tag{1.2}$$

The solution obviously satisfies the above conditions (i) and (ii).

The giant magnon solution is strikingly similar to the dark soliton [13, 14]

$$\phi = \frac{1}{2\sqrt{g}}\left[v - i\sqrt{2\mu-v^2}\tanh\frac{\sqrt{2\mu-v^2}(x-vt)}{2}\right] \tag{1.3}$$

of the NLS model:[2]

$$S_{\mathrm{NLS}} = \int dt\,dx\,\left(i\phi^*\dot{\phi} - |\acute{\phi}| - g|\phi|^4 + \mu|\phi|^2\right). \tag{1.4}$$

An apparent difference between the the giant magnon and the NLS soliton is that the size of the latter can be arbitrary large and actually becomes infinite at $v^2 = 2\mu$, while the size of the giant magnon depends on the velocity only through the trivial Lorentz-contraction factor and is always smaller than $1/\mu$. We will see in section 2 that this is an artifact of the classical approximation. The quantum giant magnon also has a variable, velocity-dependent size which turns to infinity when soliton moves at the speed of sound.

In the framework of Bethe Ansatz the ground state of the quantum NLS model at non-zero chemical potential is represented by a Fermi sea of interacting particles [15], 1d bosons with a local repulsive interaction for which (1.4) is the second-quantized action. Because of the repulsion particles in some sense obey the Fermi statistics. The spectrum of elementary excitations has two branches, the particles and the holes. At weak coupling ($g \ll \sqrt{\mu}$) the spectral properties of the hole excitations precisely match those of the classical solution (1.3), which is why the holes, at arbitrary coupling, can be interpreted as quantum dark solitons [7]. The particle branch of the spectrum interpolates between sound waves and bosonic single-particle excitations and at $g \ll \sqrt{\mu}$ is described by the Bogolyubov theory of a weakly interacting Bose gas [16, 17].

The relationship between dark solitons and holes in the Fermi sea essentially follows [7, 18] from the spectral properties of the Lax operator in the finite-density case [19]: the

[1] More precisely, this condition states that the field has the form $\langle\phi\rangle\,\mathrm{e}^{-i\mu t}$ asymptotically at spacial infinity.

[2] The time-dependent phase in ϕ here is traded for the chemical potential in the Lagrangian.

JHEP05(2008)047

spectrum of the auxiliary linear problem has a gap, which is the semiclassical counterpart of the Fermi sea in quantum theory. The dark solitons (1.3) correspond to normalizable eigenstates inside the gap and thus represent holes. The spectral density has a characteristic square-root behavior at the edges of the spectral gap. There are many instances where Bethe Ansatz reduces to singular integral equations of the matrix model type and its solutions exhibit similar square-root behavior. This happens, for example, at large N [20, 21], in the semiclassical approximation [22 – 24] or in the conformal limit [25 – 28]. We will demonstrate that Bethe equations also reduce to singular integral equations when they describe weakly interacting Bose gas. This limit of Bethe Ansatz (which we will call the Bogolyubov limit) is ubiquitous in integrable systems, and does not necessarily coincide with the classical approximation.

In particular, the large-N limit of the $O(N)$ model, in which quantum fluctuations are definitely important, falls into the category described above. Building upon this observation we will argue that quantum giant magnons should be identified with the holes in the Fermi sea. We will first construct the large-N counterpart of the classical solution (1.2) in section 2 and then compare it with the large-N limit of Bethe Ansatz in section 3. In section 4 we study the limit of small anisotropy in the XXZ spin chain which also turns out to be of the Bogolyubov type.

2. Giant magnons at large N

2.1 $O(N)$ model at finite density

In order to induce a finite density of one of the $O(N)$ charges Q_{ij} $(i, j = 1 \dots N)$ one can couple (1.1) to a chemical potential by shifting the Hamiltonian $H \to H - \mu^{ij} Q_{ij}/2$. This is equivalent to gauging the $O(N)$ symmetry by a constant A_0 and amounts to replacing ∂_0 by a covariant derivative $D_0^{ij} = \partial_0 \delta^{ij} + \mu^{ij}$ in the action. In the AdS/CFT context the finite density of the $O(6)$ charge corresponds to an infinite angular momentum uniformly distributed along the string.[3] The $N/2$ independent Cartan charges[4] are carried by complex linear combinations

$$z_I = \frac{n_{2I-1} + i n_{2I}}{\sqrt{2}}, \qquad I = 1 \dots N/2. \tag{2.1}$$

Introducing a Lagrange multiplier σ that enforces the condition $z^{*I} z_I = 1/2$, we can put (1.1) into the unconstrained form:

$$S = \frac{N}{\lambda} \int d^2x \left[\sum_{I=1}^{N/2} \left(|D_\nu z_I|^2 - \sigma |z_I|^2 \right) + \frac{1}{2}\sigma \right], \qquad D_0 z_I = \partial_0 z_I - i\mu^I z_I. \tag{2.2}$$

In principle all μ^I's are independent variables, and one can consider various combinations of the chemical potentials which give different background charges. We will be interested

[3] See [29] for a recent discussion of the canonical vs. microcanonical description of charged states in the AdS string theory

[4] For simplicity we assume that N is even. This assumption is not essential in the large-N limit.

JHEP05(2008)047

in the simplest case when only one chemical potential $\mu \equiv \mu^1$ is non-zero and the rest $\mu^I = 0$. For the field that carries the background charge we will use a special notation:[5]

$$\phi \equiv \frac{1}{\sqrt{\lambda}}\, z_1. \tag{2.3}$$

The large-N limit of the $O(N)$ model can be solved by standard methods [30]. Integrating out z_I's generates an effective action for σ, which has a minimum at a non-zero vev: $\langle\sigma\rangle|_{\mu=0} = m^2$. The vev of σ gives equal masses to all z_I fields and is determined by the gap equation:

$$\frac{1}{\lambda} = \langle x|\, \frac{i}{-\partial^2 - m^2}\, |x\rangle\,. \tag{2.4}$$

When the chemical potential is turned on it is convenient to leave the charged field ϕ unintegrated:

$$\begin{aligned} S_{\text{eff}} = N \int d^2x \left[|\partial_\nu \phi|^2 + i\mu\,(\phi^* \partial_0 \phi - \phi\, \partial_0 \phi^*) + (\mu^2 - \sigma)|\phi|^2 + \frac{1}{2\lambda}\,\sigma \right] \\ + \frac{i(N-2)}{2}\, \ln\det\left(-\partial^2 - \sigma\right). \end{aligned} \tag{2.5}$$

At large N the tree approximation for this effective action becomes exact and one can expand around the minimum of the effective potential. If $\mu > m$, the setting is the same as in the Bogolyubov theory: The zero mode of ϕ Bose condenses, with the physical ground state at

$$\langle\sigma\rangle = \mu^2, \qquad \langle\phi\rangle^2 = \frac{1}{4\pi}\, \ln\frac{\mu}{m}\,. \tag{2.6}$$

These equations are obtained by minimizing the effective action (2.5) in ϕ and σ and taking into account the dimensional transmutation formula (2.4). The value of the action at the minimum determines the density of the free energy:

$$\mathcal{E} = \frac{E_{\text{vac}}}{\text{Vol}} = -\frac{N\mu^2}{8\pi}\left(2\ln\frac{\mu}{m} - 1\right). \tag{2.7}$$

The fluctuations of ϕ around the ground state (the Bogolyubov branch of the spectrum) interpolate between phonons with $\varepsilon = c_s p$, at $p \ll \mu\ln(\mu/m)$, and single-particle excitations with $\varepsilon = p$, at $p \gg \max\{\mu\ln(\mu/m), \mu\}$. The speed of sound can be found by integrating out σ in (2.5) and linearizing the resulting equations of motion for ϕ:

$$c_s^2 = \frac{\ln\frac{\mu}{m}}{\ln\frac{\mu}{m} + 1}\,. \tag{2.8}$$

Alternatively, the same result can be obtained from the thermodynamic relation $c_s^2 = \mu^{-1}(\partial\mathcal{E}/\partial\mu)/(\partial^2\mathcal{E}/\partial\mu^2)$. In addition to sound waves, the field ϕ describes a massive mode separated from the ground state by the gap $M^2 = 8\mu^2\ln(\mu/m)$. The neutral modes have a common mass equal to μ.

[5]This is the same ϕ as in (1.2).

JHEP05(2008)047

2.2 Solitons

We now turn to the soliton sector of the large-N effective theory (2.5). The effective action (2.5) and the ensuing equations of motion are non-local:[6]

$$
\begin{aligned}
&|\phi|^2 = \frac{1}{2}\,\langle x|\left(\frac{i}{-\partial^2 - m^2} - \frac{i}{-\partial^2 - \sigma}\right)|x\rangle \\
&-\partial^2\phi + 2i\mu\,\partial_0\phi + (\mu^2 - \sigma)\phi = 0.
\end{aligned}
\tag{2.9}
$$

Similar equations arise in a variety of large-N field theories and in spite of their non-locality are solvable in some cases [31 − 33], which presumably reflects complete integrability of the underlying models. The $O(N)$ model is integrable as well and we will be able to construct the exact giant magnon solution of (2.9) by using a method [34 − 36, 33] based on the Gelfand-Dikiĭ identities [37] for the diagonal resolvent of the Sturm-Liouville operator:

$$
R\,[\mathrm{x}; V(\mathrm{x})] = \langle \mathrm{x}|\,\frac{1}{-\frac{d^2}{d\mathrm{x}^2} + V}\,|\mathrm{x}\rangle\,. \tag{2.10}
$$

With the help of the differential equation satisfied by the diagonal resolvent [37] one can prove the following remarkable identity:

$$
R\left[\mathrm{x}; \omega^2 - \frac{2\nu^2}{\cosh^2\nu\mathrm{x}}\right] = \frac{1}{2\omega} + \frac{\nu^2}{2\omega\,(\omega^2 - \nu^2)\cosh^2\nu\mathrm{x}}\,. \tag{2.11}
$$

Since the giant magnon is a traveling dispersionless wave, it is convenient to perform a Lorentz transformation to its rest frame:[7]

$$
\mathrm{x} = \frac{x^1 - vx^0}{\sqrt{1 - v^2}}\,, \qquad \mathrm{t} = \frac{x^0 - vx^1}{\sqrt{1 - v^2}}\,. \tag{2.12}
$$

and to look for solutions independent of t: $\sigma \equiv \sigma(\mathrm{x})$, $\phi \equiv \phi(\mathrm{x})$. After the Fourier transform in t, (2.9) become:

$$
\begin{aligned}
&|\phi(\mathrm{x})|^2 = \int_{-\infty}^{+\infty}\frac{d\omega}{4\pi}\,\left(R\left[\mathrm{x}; \omega^2 + m^2\right] - R\left[\mathrm{x}, \omega^2 + \sigma\right]\right) \\
&\acute{\acute{\phi}} - \frac{2i\mu v}{\sqrt{1 - v^2}}\,\acute{\phi} + (\mu^2 - \sigma)\phi = 0.
\end{aligned}
\tag{2.13}
$$

The identity (2.11) and the form of the classical Hoffman-Maldacena solution (1.2) suggest the following ansatz:

$$
\begin{aligned}
\sigma &= \mu^2 - \frac{2\nu^2}{\cosh^2\nu\mathrm{x}} \\
\phi &= \left(\frac{1}{4\pi}\,\ln\frac{\mu}{m}\right)^{1/2}\,\frac{\mu v - i\nu\sqrt{1 - v^2}\,\tanh\nu\mathrm{x}}{\sqrt{\nu^2 + (\mu^2 - \nu^2)v^2}}\,.
\end{aligned}
\tag{2.14}
$$

It is straightforward to check that the ansatz goes through the equations of motion (2.13),

[6] Here we used (2.4) to eliminate the cutoff dependence and to trade the bare coupling for the physical mass.

[7] In the presence of the background charge density the Lorentz invariance is spontaneously broken, so this transformation does not leave the equations of motion invariant.

JHEP05(2008)047

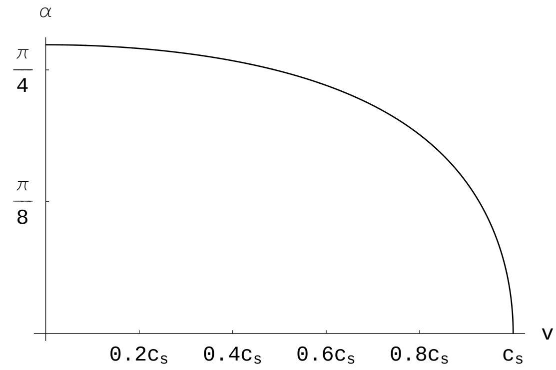

Figure 1: The inverse size of the giant magnon as a function of the velocity, at $\ln(\mu/m) = 1$. The angle α is defined in (2.15)

provided that the ratio

$$\frac{\nu}{\mu} \equiv \sin\alpha \tag{2.15}$$

satisfies

$$\alpha\left(\tan\alpha + v^2\cot\alpha\right) = \left(1 - v^2\right)\ln\frac{\mu}{m}\,. \tag{2.16}$$

The last equation determines α, and hence ν, the inverse size of the soliton, as a function of its velocity. The function $\alpha(v)$ is plotted in figure 1. It reaches its maximum at $v = 0$ and then monotonously decreases with the increase of v. In contradistinction to their classical counterparts, the large-N giant magnons cannot move faster than sound:[8] when v approaches c_s, defined in (2.8), α goes to zero. The soliton becomes larger and larger and completely dissociates when $v = c_s$.

In the weak-coupling limit, $\mu \gg m$, the large-N solution (2.14)–(2.16) goes over to the classical giant magnon (1.2), because then $\alpha \approx \pi/2$ (unless the velocity is very close to the speed of sound) and consequently $\nu \approx \mu$. An overall logarithmic factor in (2.14) arises because of the different normalization of the field ϕ, eq. (2.3). The bare coupling λ there gets replaced by the running coupling at the scale μ: $\lambda \to 2\pi/\ln(\mu/m)$.

To calculate the energy and the momentum of the giant magnon, we first compute the effective Lagrangian:

$$T\mathcal{L} = S_{\text{eff}}[\phi, \sigma] - S_{\text{vac}}. \tag{2.17}$$

The vacuum term subtracts the bulk energy and the momentum of the background state without the soliton. It is a bit surprising that the vacuum carries not only the bulk energy but also a finite amount of momentum. The non-zero momentum arises because the soliton belongs to a sector with twisted boundary conditions. The phase of ϕ experiences a non-zero increment on the soliton solution (2.14):

$$\Delta\varphi = -2\arctan\frac{\nu\sqrt{1-v^2}}{\mu v}\,. \tag{2.18}$$

[8]The classical approximation is accurate at asymptotically high densities when $\ln(\mu/m)$ is large and according to (2.8) the speed of sound approaches one. The limiting velocity thus is not visible in the classical approximation.

Consequently, the ground state must be position-dependent, in order to satisfy the boundary conditions:

$$\phi_{\rm vac} = \left(\frac{1}{4\pi}\ln\frac{\mu}{m}\right)^{1/2} {\rm e}^{\,i\Delta\varphi\sqrt{1-v^2}\,{\rm x}/L}, \tag{2.19}$$

where L is the size of the system. At $L \to \infty$ the phase changes so slowly that it does not contribute to the energy, but it still contributes a finite amount to the momentum (the momentum density due to the phase rotation is $O(1/L)$, while the energy density is $O(1/L^2)$). The details of the calculation can be found in appendix A. The result is

$$\frac{\pi}{N\mu}\,\mathcal{L}(v) = \ln\frac{\mu}{m}\,v\arctan\frac{\sqrt{1-v^2}\,\sin\alpha}{v} - \sqrt{1-v^2}\left[\left(\ln\frac{\mu}{m}-1\right)\sin\alpha + \alpha\cos\alpha\right]. \tag{2.20}$$

The energy and momentum of the giant magnon can now be found from

$$p = \frac{d\mathcal{L}}{dv}\,, \qquad \varepsilon = pv - \mathcal{L}. \tag{2.21}$$

The calculation is simplified by the fact that $\partial\mathcal{L}/\partial\alpha = 0$ as long as (2.16) is satisfied, so that $d\mathcal{L}/dv = \partial\mathcal{L}/\partial v$:

$$\frac{\pi}{N\mu}\,p = \ln\frac{\mu}{m}\,\arctan\frac{\sqrt{1-v^2}\,\sin\alpha}{v} - \frac{v\sin\alpha}{\sqrt{1-v^2}} \tag{2.22}$$

$$\frac{\pi}{N\mu}\,\varepsilon = \frac{\alpha\sec\alpha - \sin\alpha}{\sqrt{1-v^2}}\,. \tag{2.23}$$

These two equations, together with (2.16), determine the dispersion relation $\varepsilon = \varepsilon(p)$ of the giant magnon in an implicit form.

Contrary to naive expectations, the momentum of the giant magnon decreases with increasing velocity. Since $\alpha = 0$ at $v = c_s$, the magnon moving at the speed of sound has zero momentum and zero energy. As the velocity approaches zero, the momentum reaches its maximal value

$$p_F = \frac{N\mu}{2}\,\ln\frac{\mu}{m}\,, \tag{2.24}$$

which we will call the Fermi momentum for the reasons that will become clear in the next section. The energy is a periodic function of the momentum with the period $2p_F$:

$$\varepsilon(p + 2p_F) = \varepsilon(p), \tag{2.25}$$

because of the ambiguity in choosing the branch of the arctangent in (2.22). The momentum is thus naturally confined within a single "Brillouin zone" $-p_F < p < p_F$.

The Fermi momentum and the Fermi energy grow logarithmically with μ. At very large μ:

$$p_F = \frac{\pi N\mu}{\lambda(\mu)}\,, \qquad \varepsilon_F \approx \frac{2N\mu}{\lambda(\mu)} \qquad (\mu \to \infty), \tag{2.26}$$

where $\lambda(\mu) = 2\pi/\ln(\mu/m)$ is the running coupling. The limit of large chemical potential is the weak-coupling perturbative limit. The second term on the right-hand-side of (2.22) can then be neglected. Also α approaches $\pi/2$, and the dispersion relation becomes

$$\varepsilon(p) \approx \varepsilon_F \sin\frac{\pi p}{2p_F} \qquad (\mu \to \infty). \tag{2.27}$$

JHEP05(2008)047

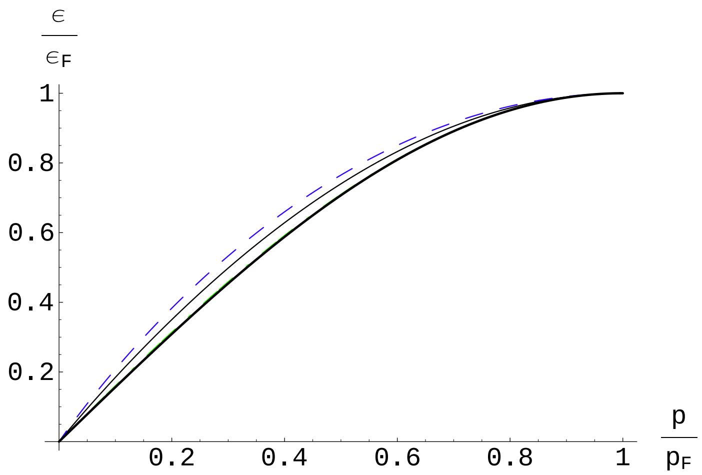

Figure 2: The normalized dispersion relation of the giant magnon: the thick solid curve is the sin law, eq. (2.27); the exact dispersion relation is practically indistinguishable from it already at $\ln(\mu/m) = 10$ (dot-dashed green line). The thin solid line corresponds to $\ln(\mu/m) = 1$ and the dashed blue line to $\ln(\mu/m) = 0.01$.

The periodicity in momentum is manifest here. In the classical approximation it has a nice geometric interpretation [1]: The momentum of the classical giant magnon is the angle subtended by the ends of the string on the sphere. It is interesting that the momentum gets non-geometric quantum corrections already in the large-N approximation. The geometric arctan term in (2.22)), of order $1/\lambda(\mu)$, is shifted by a quantum term, of order one, which has no apparent geometric meaning. Numerically, (2.27) is a good approximation in the whole range of parameters, as can be seen from figure 2.

3. Bethe Ansatz

The exact quantum spectrum of the $O(N)$ model consists of N massive particles in the vector representation of $O(N)$, whose S-matrix is known exactly at any N [8]. The ground state at finite density is the Fermi sea of fundamental particles that occupy a finite rapidity interval. The distribution of particles in the ground state at non-zero chemical potential and zero temperature is given by the solution of the following integral equation [10, 11]:

$$\varepsilon(\theta) - \int_{-B}^{B} d\xi \, K(\theta - \xi)\varepsilon(\xi) = m \cosh\theta - \mu. \tag{3.1}$$

The kernel $K(\theta)$ is the derivative of the scattering phase shift that can be extracted from the exact S-matrix by taking the matrix element responsible for scattering of particles that carry the background charge (at large N these are the quanta of the field ϕ in (2.5)). The kernel is a rather involved function of the relative rapidity [11]:

$$\begin{aligned} K(\theta) = \frac{1}{4\pi^2} \Bigg[& \psi\left(\frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{N-2} + \frac{i\theta}{2\pi}\right) + \psi\left(\frac{1}{2} + \frac{1}{N-2} + \frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right) \\ & + \psi\left(-\frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{N-2} - \frac{i\theta}{2\pi}\right) + \psi\left(\frac{1}{2} + \frac{1}{N-2} - \frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right) \Bigg]. \end{aligned} \tag{3.2}$$

The equation (3.1) describes the filling of the Fermi sea in the thermodynamic limit. The rapidity interval $(-B, B)$ is occupied, while the outside of the interval is empty. The equation describes not only the ground state, but also the spectrum of excitations.[9] The ground state energy is given by

$$\mathcal{E} = \frac{m}{2\pi} \int_{-B}^{B} d\theta \, \varepsilon(\theta) \cosh\theta. \tag{3.3}$$

The function $\varepsilon(\theta)$, often called pseudo-energy, is the energy of a particle (for $|\theta| > B$) or a hole (for $|\theta| < B$) with rapidity θ. Consequently, $\varepsilon(\theta) \lesseqgtr 0$ at $|\theta| \lesseqgtr B$. The condition that $\varepsilon(\pm B) = 0$ unambiguously determines the Fermi rapidity B.

We will be mostly interested in the hole excitations. To find their dispersion relation $\varepsilon = \varepsilon(p)$ one has to solve an additional equation [38]:

$$\acute{p}(\theta) - \int_{-B}^{B} d\xi \, K(\theta - \xi) \acute{p}(\xi) = -m \cosh\theta, \tag{3.4}$$

which determines (the derivative of) the momentum. We are going to show that the hole excitations are equivalent at $N \to \infty$ to the solitons constructed in previous section. It is instructive to consider first a much simpler case of the NLS model, where the holes in the Fermi sea can be shown to describe quantum dark solitons.

3.1 Non-linear Schrödinger model

The integral equations for the NLS model (1.4) are [15, 39, 38]

$$\varepsilon(v) - \frac{g}{\pi} \int_{-B}^{B} \frac{du \, \varepsilon(u)}{(v-u)^2 + g^2} = v^2 - \mu \tag{3.5}$$

$$\acute{p}(v) - \frac{g}{\pi} \int_{-B}^{B} \frac{du \, \acute{p}(u)}{(v-u)^2 + g^2} = -1. \tag{3.6}$$

These equations are much simpler than the equations for the ground state of the $O(N)$ model, yet they are not solvable analytically. In [15] they were analyzed numerically. Not surprisingly the equations considerably simplify in the Bogolyubov limit $g \to 0$, such that they admit an analytic solution. At first sight, the kernel simply disappears at $g \to 0$, because the scattering phase is then very small. Neglecting the kernel, however, would lead to totally misleading results, because the scattering phase is small only for $|v - u| \gg g$. If $|u - v| \sim g$ the kernel on the contrary is very large: $K \sim 1/g$. In fact, $K(v)$ approximates the delta-function at small g. But replacing $K(v)$ by $\delta(v)$ would again be wrong,[10] since then the left-hand side of (3.5) completely disappears. The correct procedure consists in

[9] The equation (3.1) does not take into account the spin degrees of freedom. The nested Bethe equations, which describe spins of the particles, are analyzed for the $O(N)$ model at finite density in [26 – 28]. It is interesting that the spin excitations have much in common with giant magnons [27].

[10] Such an approximation is correct at finite temperature or, more precisely at $T \gg \mu^{3/2}/g$, and leads to the standard Bose distribution [39].

keeping the next-to-leading $O(g)$ term:[11]

$$\frac{g}{v^2+g^2} \approx \pi\,\delta(v) + \wp\frac{g}{v^2} \qquad (g\to 0). \tag{3.7}$$

The equation for pseudo-energy of holes then reduces to a singular integral equation:

$$\frac{g}{\pi}\fint_{-B}^{B}\frac{du\,\varepsilon(u)}{(v-u)^2} = \mu - v^2 \qquad (|v|<B). \tag{3.8}$$

Integrating once we get:

$$-\frac{g}{\pi}\fint_{-B}^{B}\frac{du\,\varepsilon(u)}{v-u} = \mu v - \frac{1}{3}v^3, \tag{3.9}$$

and, similarly,

$$-\frac{g}{\pi}\fint_{-B}^{B}\frac{du\,\acute{p}(u)}{v-u} = v. \tag{3.10}$$

The energy and momentum of holes thus scale as $1/g$. Once (3.9), (3.10) are solved, the energy and momentum of particles can be found by simple integration:

$$\begin{aligned}\varepsilon(v) &= v^2-\mu+\frac{g}{\pi}\int_{-B}^{B}\frac{du\,\varepsilon(u)}{(v-u)^2} \qquad (|v|>B)\\ \acute{p} &= 1-\frac{g}{\pi}\int_{-B}^{B}\frac{du\,\acute{p}(u)}{(v-u)^2+g^2} \qquad (|v|>B).\end{aligned} \tag{3.11}$$

The last terms in these equations cannot be neglected, since inside the Fermi interval $\varepsilon(v)$ and $\acute{p}(v)$ are $O(1/g)$.

The equation (3.9) is easily solvable. It also admits an interesting interpretation in terms of random matrix theory, where such an equation arises as an equilibrium condition for an eigenvalue distribution [40], which can be pictured as a macroscopically large number of particles in an external potential $V_{\rm ext} = \mu v^2/2 - v^4/6$ subject to pairwise logarithmic repulsion. The equation itself does not determine the Fermi velocity B. In matrix models the normalization of the density $-\varepsilon(v)$ uniquely determines B [40], but here the total number of particles is not fixed and in order to find the Fermi velocity we need to minimize the free energy:

$$\mathcal{E} = \frac{1}{2\pi}\int_{-B}^{B}dv\,\varepsilon(v). \tag{3.12}$$

In the matrix-model language, $\mathcal{E}$ is the total number of particles (up to a sign since $\varepsilon(v)$ is negative inside the Fermi interval and thus $\mathcal{E}<0$). Therefore, we need to increase the number of particles as much as possible in order to minimize $\mathcal{E}$. This cannot be done indefinitely because the potential $V_{\rm ext}$ has the shape of an upside-down double well. The repulsion between the particles counteracts the attraction towards the bottom of the potential and tends to spread the particle's distribution. Eventually, if the number of particles is sufficiently large, the repulsion wins and the particles start to spill out of the

[11] This is only important for holes. For particles the delta-function is concentrated outside of the region of integration.

JHEP05(2008)047

potential well. Mathematically this means that for sufficiently large B, eq. (3.9) has no solutions with $\varepsilon(v) < 0$ within the Fermi interval $(-B, B)$. The free energy is minimized by the critical solution, when the particles are just starting to spill out of the potential well. The critical point is characterized by the change in the edge behavior of the particle's density. Normally $\varepsilon(v) \sim (B - v)^{1/2}$, but at the critical point [40]

$$\varepsilon(v) \sim (B - v)^{3/2}. \tag{3.13}$$

The criticality gives an extra condition that determines B. Imposing this condition on the solution of (3.9), we find:

$$\varepsilon(v) = \frac{1}{3g} \left(2\mu - v^2\right)^{3/2} \tag{3.14}$$

$$p(v) = \frac{\mu}{g} \arctan \frac{\sqrt{2\mu - v^2}}{v} - \frac{v}{2g} \sqrt{2\mu - v^2} \tag{3.15}$$

$$\mathcal{E} = -\frac{\mu^2}{8g}. \tag{3.16}$$

These are, respectively, the energy of the dark soliton (1.3) [7], its momentum [7], and the energy density of the ground state at $\langle\phi\rangle = \sqrt{\mu/2g}$.

3.2 $O(N)$ sigma model

The large-N limit of the Bethe Ansatz in the $O(N)$ model is very similar to the weak coupling limit for NLS. The large-N expansion of the kernel (3.2) starts with the delta-function. Keeping the next-to-leading term, we get:

$$K(\theta) \approx \delta(\theta) + \frac{1}{N} \wp \left(\frac{\cosh\theta}{\sinh^2\theta} + \frac{1}{\theta^2} \right). \tag{3.17}$$

Repeating the same steps as in the NLS case, we arrive at the singular integral equations for the pseudo-energy of holes:

$$-\frac{1}{N} \fint_{-B}^{B} d\xi\, \varepsilon(\xi) \left(\frac{1}{\theta - \xi} + \frac{1}{\sinh(\theta - \xi)} \right) = \mu\theta - m\sinh\theta, \tag{3.18}$$

and for their momentum:

$$-\frac{1}{N} \fint_{-B}^{B} d\xi\, \acute{p}(\xi) \left(\frac{1}{\theta - \xi} + \frac{1}{\sinh(\theta - \xi)} \right) = m\sinh\theta. \tag{3.19}$$

The singular integral equations with a combination of rational and hyperbolic kernels are not solvable by standard techniques, but as we will argue, the solution is implicitly given by the dispersion relation of the large-N giant magnon, eqs. (2.22), (2.23), (2.16). It is straightforward to check this perturbatively in B and θ, which is effectively an expansion in $\ln(\mu/m)$. The tricky part is to find the relationship between the rapidity θ, that enters the Bethe equations, and the velocity of the giant magnon, or the parameter α defined in (2.16). To the first few orders in B,

$$\alpha = \left(\frac{1}{2} - \frac{B^2}{24} + \frac{B^4}{144} + \frac{B^2\theta^2}{360} + \dots \right) \sqrt{B^2 - \theta^2}\,. \tag{3.20}$$

Then

$$\begin{aligned}
\varepsilon(\theta) &= -\frac{N\mu}{12\pi}\left(1-\frac{7B^2}{40}+\frac{\theta^2}{20}+\frac{527B^4}{13440}-\frac{3B^2\theta^2}{280}+\frac{\theta^4}{840}+\dots\right)\left(B^2-\theta^2\right)^{3/2} \\
p(\theta) &= \frac{N\mu}{4\pi}\left[\left(B^2-\frac{B^4}{12}+\frac{11B^6}{720}+\dots\right)\arccos\frac{\theta}{B}\right. \\
&\quad \left.-\left(\theta-\frac{B^2\theta}{6}+\frac{\theta^3}{12}+\frac{11B^4\theta}{320}-\frac{7B^2\theta^3}{320}+\frac{\theta^5}{360}+\dots\right)\sqrt{B^2-\theta^2}\right]
\end{aligned} \tag{3.21}$$

indeed solve (3.18) and (3.19) provided that

$$\frac{\mu}{m} = 1+\frac{B^2}{4}+\frac{B^4}{96}+\dots\,. \tag{3.22}$$

This perturbative solution can be pushed to any reasonable order using `Mathematica` and passes a number of consistency checks: The free energy computed from (3.3):

$$\mathcal{E} = \frac{N\mu^2}{8\pi}\left(1-\frac{B^2}{2}+\frac{B^4}{24}-\frac{11B^6}{1440}+\dots\right) \tag{3.23}$$

agrees with (2.7) upon identification (3.22).

Since $\varepsilon(\theta) \sim (B-\theta)^{3/2}$, we can differentiate (3.18) in m without the risk of producing a singularity at the edge the Fermi interval. This gives the relationship:

$$\acute{p} = -m\,\frac{\partial\varepsilon}{\partial m}\,, \tag{3.24}$$

which is also compatible with the solution (3.21).

The last equation can be used to calculate the exact Fermi rapidity. Near the Fermi point $\theta = B$, the pseudo-energy has the form $\varepsilon(\theta) = -P(m,\theta)(B-\theta)^{3/2}$, where $P(m,\theta)$ is analytic at $\theta = B$. Differentiating in m, we find from (3.24):

$$\acute{p}(\theta) = \frac{3}{2}\,mP(m,B)\,\frac{\partial B}{\partial m}\,(B-\theta)^{1/2} + O\left((B-\theta)^{3/2}\right),$$

or

$$p(\theta) = -mP(m,B)\,\frac{\partial B}{\partial m}\,(B-\theta)^{3/2} + O\left((B-\theta)^{5/2}\right).$$

The ratio ε/p at the Fermi point coincides with the speed of sound:

$$c_s = -\lim_{\theta\to B}\frac{\varepsilon(\theta)}{p(\theta)} = -\frac{1}{m\,\frac{\partial B}{\partial m}}\,. \tag{3.25}$$

Equating this to (2.8) gives a differential equation that determines B:

$$B = \sqrt{\ln\frac{\mu}{m}\left(\ln\frac{\mu}{m}+1\right)} + \operatorname{arcsinh}\sqrt{\ln\frac{\mu}{m}}\,. \tag{3.26}$$

Inverting this equation and expanding in B, we find (3.22).

JHEP05(2008)047

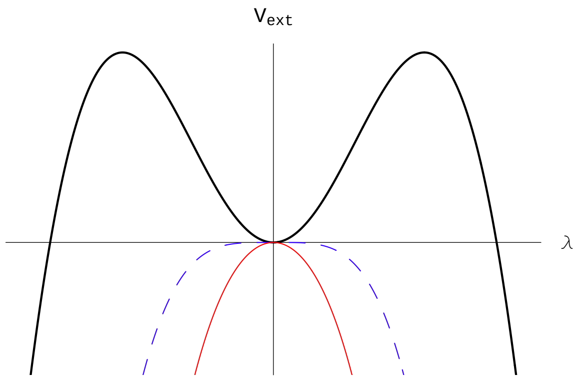

Figure 3: The effective potential for the integral equation (4.9): at $h = 0.8h_c$ (thick solid black curve); at $h = h_c$ (dashed blue curve); and at $h = 1.2h_c$ (thin solid red curve).

4. Nearly isotropic XXZ spin chain

In this section we consider the XXZ spin chain in the magnetic field:

$$H_{XXZ} = -\sum_{l=1}^{L} \left(\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y + \cos 2\eta \, \sigma_l^z \sigma_{l+1}^z + h \sigma_l^z \right) . \tag{4.1}$$

The ground state is described by the integral equations [38]

$$\varepsilon(\lambda) - \int_{-B}^{B} d\nu \, K(\lambda - \nu)\varepsilon(\nu) = \varepsilon_0(\lambda) \tag{4.2}$$

$$\acute{p}(\lambda) - \int_{-B}^{B} d\nu \, K(\lambda - \nu)\acute{p}(\nu) = -\acute{p}_0(\lambda) \tag{4.3}$$

$$\mathcal{E} = \frac{1}{2\pi} \int_{-B}^{B} d\lambda \, \acute{p}_0(\lambda)\varepsilon(\lambda), \tag{4.4}$$

where

$$K(\lambda) = \frac{\sin 4\eta}{2\pi \sinh(\lambda + 2i\eta) \sinh(\lambda - 2i\eta)} \tag{4.5}$$

$$\varepsilon_0(\lambda) = 2h - \frac{2 \sin^2 2\eta}{\cosh(\lambda + i\eta) \cosh(\lambda - i\eta)} \tag{4.6}$$

$$\acute{p}_0(\lambda) = \frac{\sin 2\eta}{\cosh(\lambda + i\eta) \cosh(\lambda - i\eta)} . \tag{4.7}$$

The limit of small anisotropy, $\eta \to 0$, and small magnetic field $h \sim \eta^2$ can be interpreted as the Bogolyubov limit. Indeed the small-η expansion of the kernel (4.5) starts with the delta function:

$$K(\lambda) \approx \delta(\lambda) + \wp \frac{2\eta}{\pi \sinh^2 \lambda} \qquad (\eta \to 0), \tag{4.8}$$

and Bethe Ansatz reduces to singular integral equations:

$$-\fint_{-B}^{B} \frac{d\nu}{\pi} \, \varepsilon(\nu) \coth(\lambda - \nu) = 4\eta \tanh \lambda - \frac{h}{\eta} \lambda. \tag{4.9}$$

$$\fint_{-B}^{B} \frac{d\nu}{\pi} \, \acute{p}(\nu) \coth(\lambda - \nu) = -\tanh \lambda. \tag{4.10}$$

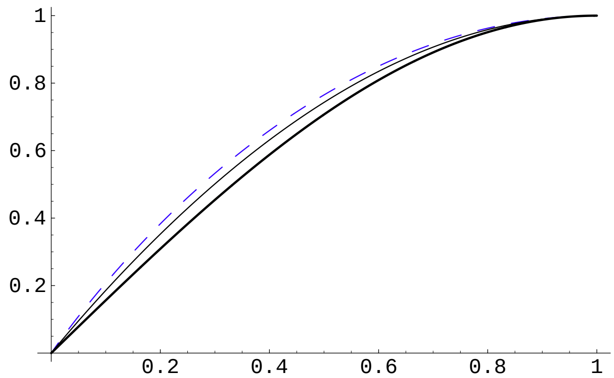

Figure 4: The dispersion relation for dark soliton in the XXZ spin chain for various values of the magnetic field: $h_c/h = 100$ (green dot-dashed line); $h_c/h = 2$ (thin solid line) and $h_c/h = 1.01$ (dashed blue line). It is accurately fitted by a simple dispersion law (2.27) shown in thin black line.

The effective "matrix-model" potential in (4.9), $V_{\rm ext} = 4\eta \ln\cosh\lambda - h\lambda^2/2\eta$, has a stable minimum only if $h < 4\eta^2$. At the critical magnetic field $h_c = 4\eta^2$ the minimum disappears (figure 3), the Fermi interval shrinks to a point, and for $h > h_c$ the equation(4.9) has no solutions with negative pseudo-energy. The ground state at a supercritical magnetic field is the completely empty ferromagnetic vacuum.

The coth kernel in (4.9) can be explicitly inverted. After straightforward albeit lengthy calculations we find the solution to (4.9) at criticality:

$$\varepsilon(\lambda) = -\frac{1}{\eta\cosh\lambda}\sqrt{16\eta^4 - h^2\cosh^2\lambda} + \frac{h}{2\eta}\,\arccos\left(\frac{h^2}{8\eta^4}\cosh^2\lambda - 1\right), \tag{4.11}$$

where the Fermi point is given by

$$\cosh B = \frac{4\eta^2}{h}\,.$$

One can verify that the pseudo-energy is negative everywhere in the interval $(-B, B)$ and behaves as $|B \pm \lambda|^{3/2}$ at the edges.

From (4.4) we get for the free energy density:

$$\mathcal{E} = -\frac{(4\eta^2 - h)^2}{8\eta^2} = -\frac{(h_c - h)^2}{2h_c}\,. \tag{4.12}$$

The momentum can be computed by noticing that

$$\acute{p} = \frac{\partial\,(\varepsilon\eta)}{\partial h_c}\,, \tag{4.13}$$

which gives:

$$p(\lambda) = \arccos\frac{h_c\tanh\lambda}{\sqrt{h_c^2 - h^2}} - \frac{h}{h_c}\,\arccos\frac{h\sinh\lambda}{\sqrt{h_c^2 - h^2}}\,. \tag{4.14}$$

The velocity of sound is

$$c_s = -\lim_{\lambda\to B}\frac{\varepsilon(\lambda)}{p(\lambda)} = \frac{\sqrt{h_c^2 - h^2}}{\eta}\,. \tag{4.15}$$

JHEP05(2008)047

The dispersion relation is shown in figure 4 and is well approximated by (2.27), especially for small values of the magnetic field.

The energy of a hole (which can presumably be identified with some sort of a soliton) is periodic in momentum with the period $2p_F$, where the Fermi momentum is given by

$$p_F = \frac{\pi}{2}\,\frac{h_c - h}{h_c}\,. \tag{4.16}$$

For very small magnetic fields the period is just the inverse of the lattice spacing (= 1). This is not surprising, since at zero magnetic field the hole excitations are magnons of the XXZ spin chain. Periodicity of their momentum is a consequence of the underlying lattice structure. However, the effective lattice spacing, $a_{\rm eff} = h_c/(h_c - h)$, grows with the magnetic field and becomes infinite at the critical point. The periodicity in momentum should have some other origin near the critical point, not related to the lattice structure of the spin chain.

5. Conclusions

The giant magnons in the $O(N)$ sigma-model, as well as other dark soliton in integrable theories, can be identified with the holes in the Fermi sea. The mysterious periodicity of their momentum has a rather mundane explanation from this point of view — the period is just the Fermi momentum doubled. It is not clear what implications can have such an interpretation for the AdS/CFT correspondence. Unlike the string sigma-model, the $O(N)$ model is not conformal, it is a massive field theory with non-zero beta-function and dimensional transmutation. In addition, the string sigma-model is coupled to 2d gravity and one should fix the diffeomorphism gauge and solve or impose the Virasoro constraints.[12] This eliminates longitudinal degrees of freedom, which in the $O(N)$ model correspond to the Bogolyubov sound waves. The giant magnons, however are transverse since they satisfy the Virasoro constraints [1] at least classically.

In string theory, the finite charge density arises when a physical gauge condition of light-cone type is imposed. The zero-density state and the spectrum of excitations around it presumably correspond to the covariant, conformal-gauge description of the sigma-model on $AdS_5 \times S^5$, which at the moment is not developed to the degree that one could formulate Bethe Ansatz in the bare vacuum.

Acknowledgments

I would like to thank G. Arutyunov, N. Gromov, V. Kazakov, C. Kristjansen, Yu. Makeenko, J. Minahan, O. Ohlsson Sax, F. Smirnov, P. Vieira and A. Zamolodchikov for interesting discussions and the Isaac Newton Institute for Mathematical Sciences for kind hospitality during the programme "Strong Fields, Integrability and Strings". This work was supported by the Knut and Alice Wallenberg Foundation through the Royal Swedish

[12] See [28] for a discussion of the Virasoro constraints from the Bethe-Ansatz point of view in the conformal limit of the $O(N)$ model.

Academy of Sciences Research Fellowship, in part by the Swedish Research Council under contract 621-2007-4177, and in part by the Isaac Newton Institute for Mathematical Sciences.

A. Action of the giant magnon

Here we compute the action of the large-N giant magnon. All the "classical" terms in the effective action (2.5), those that depend on ϕ, do not contribute since upon integration by parts they yield the equation of motion for ϕ, of which the giant magnon is a solution. The easiest way to compute the "quantum" part of the action is to differentiate it in ν:

$$\begin{aligned}\frac{\partial S}{\partial \nu} &= \frac{N}{2}\int d^2x\, \frac{\partial \sigma}{\partial \nu}\left(\frac{1}{\lambda} - \langle x|\,\frac{i}{-\partial^2-\sigma}\,|x\rangle\right)\\ &= TN\sqrt{1-v^2}\int d\mathrm{x}\,\frac{\partial\sigma}{\partial\nu}\int_{-\infty}^{+\infty}\frac{d\omega}{4\pi}\,\left(R\left[\mathrm{x};\omega^2+m^2\right]-R\left[\mathrm{x};\omega^2+\sigma\right]\right)\\ &= \frac{1}{\pi}\left(\alpha\tan\alpha - \ln\frac{\mu}{m}\right),\end{aligned}$$

where in the last line we used (2.11) and the explicit form of the solution (2.14). Requiring that $S(\nu=0)=0$ effectively subtracts the background energy (but not the background momentum!), and yields:

$$S = -\frac{1}{\pi}\,TN\mu\sqrt{1-v^2}\left[\left(\ln\frac{\mu}{m}-1\right)\sin\alpha+\alpha\cos\alpha\right].$$

The background action due to the phase rotation in (2.19) is

$$S_{\rm vac} = \frac{1}{2\pi}\,TN\mu v\,\Delta\varphi\,\ln\frac{\mu}{m}\,.$$

Subtracting $S_{\rm vac}$ from S, and using (2.18), we get (2.20).

References

[1] D.M. Hofman and J.M. Maldacena, *Giant magnons*, *J. Phys.* **A 39** (2006) 13095 [hep-th/0604135].

[2] J.A. Minahan, *Zero modes for the giant magnon*, *JHEP* **02** (2007) 048 [hep-th/0701005].

[3] G. Papathanasiou and M. Spradlin, *Semiclassical quantization of the giant magnon*, *JHEP* **06** (2007) 032 [arXiv:0704.2389].

[4] H.-Y. Chen, N. Dorey and R.F. Lima Matos, *Quantum scattering of giant magnons*, *JHEP* **09** (2007) 106 [arXiv:0707.0668].

[5] N. Gromov, S. Schäfer-Nameki and P. Vieira, *Quantum wrapped giant magnon*, arXiv:0801.3671.

[6] M.P. Heller, R.A. Janik and T. Lukowski, *A new derivation of Lüscher F-term and fluctuations around the giant magnon*, arXiv:0801.4463.

[7] P.P. Kulish, S.V. Manakov and L.D. Faddeev, *Comparison of the exact quantum and quasiclassical results for the nonlinear Schrödinger equation*, *Theor. Math. Phys.* **28** (1976) 615 [*Teor. Mat. Fiz.* **28** (1976) 38].

[8] A.B. Zamolodchikov and A.B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, *Ann. Phys. (NY)* **120** (1979) 253.

[9] A.M. Polyakov and P.B. Wiegmann, *Theory of nonabelian Goldstone bosons in two dimensions*, *Phys. Lett.* **B 131** (1983) 121.

[10] P. Hasenfratz, M. Maggiore and F. Niedermayer, *The Exact mass gap of the $O(3)$ and $O(4)$ nonlinear σ-models in $D = 2$*, *Phys. Lett.* **B 245** (1990) 522.

[11] P. Hasenfratz and F. Niedermayer, *The Exact mass gap of the $O(N)$ σ-model for arbitrary $N \geqslant 3$ in $D = 2$*, *Phys. Lett.* **B 245** (1990) 529.

[12] Y.S. Kivshar and B. Luther-Davies, *Dark optical solitons: physics and applications*, *Phys. Rept.* **298** (1998) 81.

[13] T. Tsuzuki, *Nonlinear waves in the Pitaevskii-Gross equation*, *J. Low Temp. Phys.* **4** (1971) 441.

[14] V.E. Zakharov and A.B. Shabat, *Interaction between solitons in a stable medium*, *Sov. Phys. JETP* **37** (1973) 823.

[15] E.H. Lieb and W. Liniger, *Exact analysis of an interacting Bose gas. 1. The General solution and the ground state*, *Phys. Rev.* **130** (1963) 1605.

[16] N.N. Bogolyubov, *On the theory of superfluidity*, *Izv. Akad. Nauk SSSR Ser. Fiz.* **11** (1947) 77.

[17] A.A. Abrikosov, L.P. Gorkov and I.E. Dzyaloshinski, *Methods of quantum field theory in statistical physics*, Prentice-Hall (1963).

[18] F.A. Smirnov, *Connection between the sine-Gordon model and the massive Bose-thirring model*, *Theor. Math. Phys.* **53** (1982) 1153 [*Teor. Mat. Fiz.* **53** (1982) 323].

[19] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer-Verlag (1987).

[20] V.A. Fateev, V.A. Kazakov and P.B. Wiegmann, *Large N chiral field in two-dimensions*, *Phys. Rev. Lett.* **73** (1994) 1750.

[21] V.A. Fateev, V.A. Kazakov and P.B. Wiegmann, *Principal chiral field at large-N*, *Nucl. Phys.* **B 424** (1994) 505 [hep-th/9403099].

[22] B. Sutherland, *Low-lying eigenstates of the one-dimensional Heisenberg ferromagnet for any magnetization and momentum*, *Phys. Rev. Lett.* **74** (1995) 816.

[23] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *JHEP* **09** (2003) 010 [hep-th/0306139].

[24] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[25] N. Mann and J. Polchinski, *Bethe ansatz for a quantum supercoset σ-model*, *Phys. Rev.* **D 72** (2005) 086002 [hep-th/0508232].

[26] N. Gromov, V. Kazakov, K. Sakai and P. Vieira, *Strings as multi-particle states of quantum σ-models*, *Nucl. Phys.* **B 764** (2007) 15 [hep-th/0603043].

[27] N. Gromov and V. Kazakov, *Asymptotic Bethe ansatz from string σ-model on $S^3 \times R$*, *Nucl. Phys.* **B 780** (2007) 143 [hep-th/0605026].

[28] N. Gromov, V. Kazakov and P. Vieira, *Classical limit of quantum σ-models from Bethe ansatz*, PoS(SOLVAY)005 [hep-th/0703137].

[29] R. Roiban and A.A. Tseytlin, *Spinning superstrings at two loops: strong-coupling corrections to dimensions of large-twist SYM operators*, arXiv:0712.2479.

[30] M. Moshe and J. Zinn-Justin, *Quantum field theory in the large-N limit: a review*, *Phys. Rept.* **385** (2003) 69 [hep-th/0306133].

[31] R.F. Dashen, B. Hasslacher and A. Neveu, *Semiclassical bound states in an asymptotically free theory*, *Phys. Rev.* **D 12** (1975) 2443.

[32] S.-S. Shei, *Semiclassical bound states in a model with chiral symmetry*, *Phys. Rev.* **D 14** (1976) 535.

[33] J. Feinberg, *All about the static fermion bags in the Gross-Neveu model*, *Ann. Phys. (NY)* **309** (2004) 166 [hep-th/0305240].

[34] J. Feinberg, *Nontrivial saddle points and band structure of bound states of the two-dimensional $O(N)$ vector model*, *Nucl. Phys.* **B 433** (1995) 625 [hep-th/9404143].

[35] Y. Makeenko, *Exact multiparticle amplitudes at threshold in large-N component ϕ^4 theory*, *Phys. Rev.* **D 50** (1994) 4137 [hep-ph/9404312].

[36] J. Feinberg, *On kinks in the Gross-Neveu model*, *Phys. Rev.* **D 51** (1995) 4503 [hep-th/9408120].

[37] I.M. Gelfand and L.A. Dikii, *Asymptotic behavior of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-De Vries equations*, *Russ. Math. Surveys* **30** (1975) 77 [*Usp. Math. Nauk* **30** (1975) 67].

[38] V.E. Korepin, N.M. Bogolyubov and A.G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, Cambridge U.K. (1993).

[39] C.-N. Yang and C.P. Yang, *Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction*, *J. Math. Phys.* **10** (1969) 1115.

[40] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar diagrams*, *Commun. Math. Phys.* **59** (1978) 35.

JHEP05(2008)047